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Records and 2-block records of 1-dependent stationary sequences under local dependence

by

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ABSTRACT. – We first study the records of several examples of 1-dependent stationary sequences $\{X_n\}$, some of them which satisfy a local independence condition introduced in Haiman (1987a), whereas the other do not satisfy this condition.

For the second category (called with local dependence), we then extend a result of the above cited paper by proving, (under some regularity conditions on the joint distribution of (X_1, X_2, X_3)), that the 2-block record times of $\{X_n\}$ a.s. coincide, (via a translation of the time index and from some index on) with the record times of an appropriately constructed i.i.d. sequence $\{\hat{X}_n\}$, whereas the 2-block record values of $\{X_n\}$ and the record values of $\{\hat{X}_n\}$ are imbricate. © Elsevier, Paris

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RÉSUMÉ. – On étudie d'abord les records pour plusieurs exemples de suites stationnaires 1-dépendantes $\{X_n\}$, certaines satisfaisant une condition d'indépendance locale introduite dans Haiman (1987a) et d'autres ne la satisfaisant pas. Pour la deuxième catégorie nous étendons un résultat de l'article précité en démontrant (sous des conditions de régularité portant sur la loi de (X_1, X_2, X_3)) que les instantes de records par blocs de deux de $\{X_n\}$ coïncident presque sûrement (modulo une translation d'indice et à partir d'un certain rang) avec les instants de records d'une suite i.i.d. $\{\hat{X}_n\}$ convenablement construite, alors que les valeurs de records par blocs de deux de $\{X_n\}$ et les records de $\{\hat{X}_n\}$ sont imbriqués.

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1. INTRODUCTION

Let $\{X_n\}_{n\geq 1}$, be a strictly stationary sequence of random variables (r.v.'s) with continuous marginal d.f. $F(x) = P\{X_1 \leq x\}$. Many papers have been devoted to the study of the records of $\{X_n\}_{n\geq 1}$, especially when $\{X_n\}_{n\geq 1}$ is i.i.d.. For an overview of these results, the reader is referred to Nevzorov (1987) and the references cited therein.

The classical formal definition of record times T_n and record values θ_n is:

$$T_1 = 1, \ \theta_1 = X_1,$$

and for $n \geq 1$,

$$T_{n+1} = \inf\{k > T_n, \ X_k > \theta_n\} \ \text{ and } \ \theta_{n+1} = X_{T_{n+1}}.$$

However, in the sequel we will use the following more general definition of records with respect to an initial threshold θ_0, $\alpha \leq \theta_0 < \omega$, where $\alpha = \inf\{x; F(x) > 0\}$ and $\omega = \sup\{x; F(x) < 1\}$, (the left and right end points of F).

DEFINITION 1.1. – (Haiman (1987a))

$$T_1 = \inf\{k \geq 1: X_k > \theta_0\}, \ \ \theta_1 = X_{T_1}$$

and for $n \geq 1$ (1.1)

$$T_{n+1} = \inf\{k > T_n, \ X_k > \theta_n\} \ \text{ and } \ \theta_{n+1} = X_{T_{n+1}}.$$

This definition is justified by the following proposition:

PROPOSITION 1.1. – (Haiman (1987a))

If $\{(T'_n, \theta'_n)\}$ *is any other sequence such that for* $n \geq n_0$

$$T'_{n+1} = \inf\{k > T'_n,\ X_k > \theta'_n\} \text{ and } \theta'_{n+1} = X_{T'_{n+1}},$$

then there exist n_1 *and* q *such that for* $n \geq n_1$, *we have*

$$T'_n = T_{n-q} \text{ and } \theta'_n = \theta_{n-q}.$$

For m-dependent sequences (i.e. for any $t \geq 1$, $\sigma(\dots X_t)$ and $\sigma(X_{t+m-1}, \dots)$ are independent), the following theorem was obtained in (Haiman (1987a)), under the additional "local independence" hypothesis: There exists a $\beta > 0$ such that for $2 \leq k \leq m+1$

$$\limsup_{z \nearrow \omega} \left\{ \sup_{\substack{z \leq u \\ z \leq v}} P\{X_1 > u | X_k = v\}(P\{X_1 > u\})^{-\beta} \right\} < \infty. \tag{1.2}$$

THEOREM 1.1. – *Let* $\{X_n\}_{n\geq 1}$ *satisfy hypothesis (1.2). Then there exists a probability space which carries, in addition to* $\{X_n\}_{n\geq 1}$, *an i.i.d. sequence* $\{\hat{X}_n\}_{n\geq 1}$ *having the same marginal distribution and such that, if* $\{(\hat{T}_n, \hat{\theta}_n)\}_{n\geq 1}$ *are the records of* $\{\hat{X}_n\}_{n\geq 1}$, *then a.s. there exist two r.v.'s* N_0 *and* Q *such that for* $n \geq N_0$,

$$T_n = \hat{T}_{n-Q} \text{ and } \theta_n = \hat{\theta}_{n-Q}. \tag{1.3}$$

The method of proof of theorem 1.1 was adapted in order to obtain the same statement (1.3) for some classes of Gaussian stationary sequences (Haiman (1987 b) and Haiman and Puri (1993)) and also for some Markov sequences (Haiman, Kiki and Puri (1994)). A multivariate version of theorem 1.1 was obtained in Haiman (1992).

Let now $\{X_n\}_{n\geq 1}$ be any sequence of r.v.'s for which the record times T_n defined in (1.1) are a.s. finite for any $n \geq 1$. Let θ_n be the corresponding record values. Consider another sequence $\{\hat{X}_n\}_{n\geq 1}$ whose records $(\hat{T}_n, \hat{\theta}_n)$ are similarly defined a.s. for any $n \geq 1$. Let

$$M_n = \max(X_1, \dots, X_n)$$

and

$$\hat{M}_n = \max(\hat{X}_1, \dots, \hat{X}_n),\ n \geq 1.$$

It is clear that the sequences $\{(T_n, \theta_n)\}_{n\geq 1}$ and $\{(\hat{T}_n, \hat{\theta}_n)\}_{n\geq 1}$ completely determine the sequences $\{M_n\}_{n\geq 1}$ and $\{\hat{M}_n\}_{n\geq 1}$. Then, elementary arguments based on the monotonicity of the sequences $\{\theta_n\}_{n\geq 1}$ and $\{\hat{\theta}_n\}_{n\geq 1}$ lead to the following theorem 1.2 (we improve here the statements of Haiman (1992), th. 1.1, p.147 and Haiman and Puri (1993), th.1.1, p.88):

THEOREM 1.2. – *If there exist r.v.'s N_0 and Q such that a.s. for $n \geq N_0$, (1.3) is satisfied, then there exists a random variable N_1 such that for $n \geq N_1$, we have*

$$M_n = \hat{M}_n \ \ a.s. \tag{1.4}$$

It may be seen (Haiman (1987a), p. 437) that the proof of theorem 1.1 in the general case ($m \geq 1$) easily follows from the proof in the case $m = 1$, where (1.2) becomes

$$\limsup_{z \nearrow \omega} \left\{ \sup_{\substack{z<u \\ z<v}} P\{X_1 > u | X_2 = v\}(P\{X_1 > u\})^{-\beta} \right\} < \infty. \tag{1.5}$$

It also may be seen that for $m = 1$ condition (1.5) may be replaced by the weaker condition: there exists $\beta > 0$ and $k > 0$ such that

$$\limsup_{u \nearrow \omega} \left\{ \sup_{u<v<\varphi(u)} P\{X_1 > u | X_2 = v\}(P\{X_1 > u\})^{-\beta} \right\} < \infty \tag{1.6}$$

where $\varphi(u)$ is the solution of the equation

$$1 - F(\varphi(u)) = (1 - F(u))^{1+k}.$$

This is a consequence of the construction method used in Haiman (1987a) and the fact (see Haiman and Puri (1993), lemma 3.3, p. 122) that when $\{X_n\}_{n\geq 1}$ is i.i.d., we have, for any $k > 0$,

$$P\{1 - F(\theta_{n+1}) < (1 - F(\theta_n))^k \ \ i.o.\} = 0.$$

The paper is constructed as follows: in section 2, we study the records of several examples of 1-dependent sequences $\{X_n\}_{n\geq 1}$, some of them which satisfy condition (1.6) whereas the other do not satisfy this condition. We call sequences belonging to the second category "with local dependence" and show (example 7) that the records of such sequences may behave very different from the records of i.i.d. sequences.

In section 3, we consider sequences which do not necessarly satisfy condition (1.6). We show that under some regularity conditions (satisfied by all our examples) a general version of theorem 1.1 (theorem 3.1), may be obtained.

2. STUDY OF EXAMPLES

Let $(\mathrm{E}, \mathcal{E})$ be some measurable space and $f : \mathrm{E}^2 \to \mathbb{R}$ some measurable function. The most classical examples of 1-dependent stationary sequences are of the form

$$X_n = f(\xi_n, \xi_{n+1}), \quad n \geq 1, \tag{2.1}$$

where $\{\xi_n\}_{n\geq 1}$ is a sequence of i.i.d. $(\mathrm{E}, \mathcal{E})$-valued random elements.

Our examples are all of this form. However, "exotic" examples which do not have a representation like (2.1), (for a recent discussion on this subject, see for example, Matus (1993)) may be given.

The following examples 1-4 satisfy condition (1.6).

2.1. Example 1

Let $\{Y_n\}_{n\geq 1}$ be a sequence of independent $\mathcal{N}(0, 1)$ normally distributed r.v.'s and let

$$X_n = aY_n + bY_{n+1}, \quad n \geq 1, \tag{2.2}$$

where a and b are two constants such that $0 < |a| < 1$ and $a^2 + b^2 = 1$. It may be shown (see Mayeur (1996)) that if $\rho = \mathrm{Cov}(X_n, X_{n+1}) = ab < 0$, then $\{X_n\}_{n\geq 1}$ satisfies condition (1.5) whereas if $\rho > 0$, it does not satisfy this condition but satisfies condition (1.6) for any $k > 0$.

2.2. Examples 2, 3 and 4

Let $\{U_n\}_{n\geq 1}$ be a sequence of i.i.d. uniformly on $[0, 1]$ distributed r.v.'s. We now consider examples of the form $X_n = \varphi(U_n, U_{n+1})$ and show that these examples satisfy the following condition, which in turn implies (1.6): There exists $\beta > 0$ such that

$$\limsup_{u \nearrow \omega} \left\{ \sup_{u < v < \omega} P\{X_1 > u | X_2 = v\} (P\{X_1 > u\})^{-\beta} \right\} < \infty. \tag{2.3}$$

2.2.1. Example 2. – $X_n = U_n + U_{n+1}, \quad n \geq 1.$

By elementary calculations, we obtain

$$P\{X_1 \leq x\} = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2/2 & \text{if } 0 \leq x \leq 1 \\ -x^2/2 + 2x - 1 & \text{if } 1 \leq x \leq 2 \\ 1 & \text{if } x \geq 2 \end{cases} \tag{2.4}$$

and, for $1 \leq u \leq v \leq 2$,

$$P\{X_1 \leq u, X_2 \leq v\} = (u-1) + u(v-u) - \frac{1}{2}(v-1)^2 + \frac{1}{2}(u-1)^2 + uv(2-v) - \frac{1}{2}(u+v)[1-(v-1)^2] + \frac{1}{3}[1-(v-1)^3]. \tag{2.5}$$

Note that since all our examples are of the form $X_n = \varphi(U_n, U_{n+1})$, we have

$$\begin{aligned} P\{X_1 \leq u, X_2 \leq v\} &= \int_0^1 P\{\varphi(U_1, U_2) \leq u, \varphi(U_2, U_3) \leq v | U_2 = x\}\, dx \\ &= \int_0^1 P\{\varphi(U_1, x) \leq u\} \times P\{\varphi(x, U_3) \leq v\}\, dx. \end{aligned} \tag{2.6}$$

Combining (2.4) and (2.5), for $1 \leq u \leq v \leq 2$, we obtain

$$\begin{aligned} P\{X_1 > u | X_2 = v\} &= \frac{d}{dv} P\{X_1 > u, X_2 < v\} \Big/ \frac{d}{dv} P\{X_2 < v\} \\ &= \frac{1}{2}(2+v) - u. \end{aligned}$$

Next, if we put $u = 2-\varepsilon$, $v = 2-\eta$, $0 \leq \eta \leq \varepsilon \leq 1$, (here $\omega = 2$), we get

$$\frac{P\{X_1 > u | X_2 = v\}}{(P\{X_1 > u\})^\beta} = 2^\beta \frac{\varepsilon - \frac{1}{2}\eta}{\varepsilon^{2\beta}} \tag{2.7}$$

which is bounded by $2^{1/2}$ for $\beta = \frac{1}{2}$. Thus, (2.3) is satisfied.

2.2.2. Example 3. – $X_n = U_n \times U_{n+1}$, $n \geq 1$.

We obtain

$$P\{X_1 \leq x\} = \begin{cases} 0 & \text{if } x \leq 0 \\ x(1-\log x) & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

and for $0 \leq u \leq v \leq 1$,

$$P\{X_1 \leq u, X_2 \leq v\} = u + u(\log v - \log u) + u - uv.$$

Thus, if we put $u = 1 - \varepsilon$, $v = 1 - \eta$, $0 < \eta < \varepsilon < 1$, (here $\omega = 1$), we get

$$\frac{P\{X_1 > u | X_2 = v\}}{(P\{X_1 > u\})^\beta} = \frac{-\log v + u - \frac{u}{v}}{(1 - u + u\log u)^\beta(-\log v)}$$
$$\sim 2^\beta \frac{\varepsilon - \frac{1}{2}\eta}{\varepsilon^{2\beta}}, \quad \varepsilon, \quad \eta \to 0,$$

which coincides with the last term in (2.7).

2.2.3. Example 4. – $X_n = \inf(U_n, U_{n+1})$, $n \geq 1$.

We obtain

$$P\{X_1 \leq x\} = \begin{cases} 0 & \text{if } x \leq 0 \\ 2x - x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

and for $0 \leq u \leq v \leq 1$,

$$P\{X_1 \leq u, X_2 \leq v\} = u(1 - u) + uv(2 - v).$$

Thus, if we put $u = 1 - \varepsilon$, $v = 1 - \eta$, $0 < \eta < \varepsilon < 1$, we get

$$\frac{P\{X_1 > u | X_2 = v\}}{(P\{X_1 > u\})^\beta} = \frac{1 - v - u + uv}{(1 - 2u + u^2)^\beta(1 - v)} = \frac{1}{\varepsilon^{2\beta - 1}}$$

which equals 1 if $\beta = \frac{1}{2}$. Thus (2.3) is satisfied.

The following examples 5, 6 and 7 do not satisfy condition (1.6).

2.3. Example 5

Let again $\{U_n\}_{n \geq 1}$ be a sequence of i.i.d. uniformly on $[0, 1]$ distributed r.v.'s. Let

$$X_n = \max(U_n, U_{n+1}), \quad n \geq 1. \tag{2.8}$$

We obtain

$$P\{X_1 \leq x\} = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 1, \end{cases}$$

and

$$P\{X_1 \leq u, X_2 \leq v\} = u^2 v, \quad 0 \leq u \leq v \leq 1.$$

Thus

$$\frac{P\{X_1 > u, X_2 > u\}}{P\{X_1 > u\}} = \frac{1 - 2u^2 + u^3}{1 - u^2} = 1 - \frac{u^2}{1+u} \to \frac{1}{2}, \quad u \to 1.$$

Condition (1.6) is not satisfied since it implies

$$\lim_{u\to\infty} \frac{P\{X_1 > u, X_2 > u\}}{P\{X_1 > u\}} = 0. \tag{2.9}$$

Remark 2.1. – In this example, the records of $\{X_n\}_{n\geq 1}$ coincide with the records of $\{U_{n+1}\}_n$. Thus, the sequence $\{U_{n+1}\}_n$ trivially satisfies the requirements of $\{\hat{X}_n\}_{n\geq 1}$ in theorem 1.1. This shows that condition (1.6) in this theorem is sufficient but not necessary. (See also remark 3.5).

2.4. Example 6

Let $\{Z_n\}_{n\geq 0}$ be a sequence of i.i.d. exponentially with parameter 1 distributed r.v.'s and $\{J_n\}_{n\geq 1}$ a sequence of i.i.d. Bernoulli $\mathcal{B}(1, p)$ r.v.'s. We suppose $\{Z_n\}_{n\geq 0}$ and $\{J_n\}_{n\geq 1}$ independent. Let

$$X_n = \begin{cases} Z_n & \text{if } J_n = 0 \\ Z_{n-1} & \text{if } J_n = 1. \end{cases} \tag{2.10}$$

The sequence $\{X_n\}_{n\geq 1}$ is of the form (2.1), hence it is stationary 1-dependent.

We obtain

$$F(x) = P\{X_1 \leq x\} = 1 - e^{-x}$$

and, for $x_1,\ x_2 \geq 0$,

$$P\{X_1 \leq x_1, X_2 \leq x_2\} = (1-q)F(x_1)F(x_2) + q\psi(x_1, x_2).$$

where $q = p(1-p)$ and $\psi(x_1, x_2) = 1 - e^{-\inf(x_1, x_2)}$.

Thus,

$$\lim_{u\to\infty} \frac{P\{X_1 > u, X_2 > u\}}{P\{X_1 > u\}} = \lim_{u\to\infty} \frac{qe^{-u} + (1-q)e^{-2u}}{e^{-u}} = q > 0.$$

2.5. Example 7

Let $\{Z_n\}_{n\geq 0}$ and $\{J_n\}_{n\geq 1}$ be as in example 6 and let $\{V_n\}_{n\geq 1}$ be a sequence of i.i.d. uniformly on $[0, 1]$ distributed r.v.'s, independent of $\{Z_n\}_{n\geq 0}$ and $\{J_n\}_{n\geq 1}$. Let

$$X_n = \begin{cases} Z_n & \text{if } J_n = 0 \\ Z_{n-1} + V_n & \text{if } J_n = 1. \end{cases} \tag{2.11}$$

We then obtain (for further details, see Mayeur (1996), p. 47), for $x \geq 1$,

$$P\{X_1 > x\} = e^{-x}(1 + p(e-2)),$$

and

$$P\{X_1 > x, X_2 > x\} = e^{-x}p(1-p) + e^{-2x}[(1+p(e-2))^2 - p(1-p)(e-1)].$$

Thus

$$\lim_{x \to \infty} \frac{P\{X_1 > x, X_2 > x\}}{P\{X_1 > x\}} = \frac{p(1-p)}{1+p(e-2)} > 0.$$

Remark 2.2. – The record values $\{\theta_n\}$ of $\{X_n\}$ in example 6 behave as the record values of an i.i.d. sequence of exponential $E(1)$ r.v.'s (i.e. $\theta_{n+1} - \theta_n$, $n \geq 1$ form a sequence of i.i.d. exponential $E(1)$ r.v.'s, (see Nevzorov (1987))). As $n \to \infty$, the number $N(n)$ of records in $X_1, \ldots, X_n$ is approximately the same as in a sample of $np(1-p)$ i.i.d. r.v.'s (i.e. $\mathbb{E}(N) \sim \log np(1-p)$, $n \to \infty$).

Remark 2.3. – The record times $\{T_n\}$ of $\{X_n\}$ in example 7 have the particularity that

$$P\{T_{n+1} - T_n = 1 \; i.o.\} = 1, \tag{2.12}$$

whereas the record times of an i.i.d. sequence are such that

$$P\{T_{n+1} - T_n \nearrow \infty\} = 1. \tag{2.13}$$

However, an easy consequence of theorem 3.1 (section 3) is that, for any $k \geq 2$, we have

$$P\{T_{n+1} - T_n = k \; i.o.\} = 0. \tag{2.14}$$

The proof of (2.12) follows from the fact (see also remark 2.2) that

$$\begin{aligned} P\{(J_{T_n^Z} = 0, J_{T_n^Z+1} = 1, J_{T_{n+1}^Z} = 0, J_{T_{n+1}^Z+1} = 1) \\ \cap (\theta_{n+1}^Z - \theta_n^Z > 1) \; i.o.\} = 1, \end{aligned} \tag{2.15}$$

where $\{(T_n^Z, \theta_n^Z)\}$ is the sequence of records of $\{Z_n\}_{n \geq 1}$ in (2.11). Note that when an event inside the brackets of (2.15) occurs, there is a $k \geq 1$ such that $T_k = T_{n+1}^Z$ and $T_{k+1} - T_k = 1$. For further details, see Mayeur (1996), p. 49.

Remark 2.4. – Observe that in examples 5, 6 and 7, we have

$$\lim_{x\to\omega}\frac{P\{X_1>u, X_2>u\}}{P\{X_1>u\}}=\alpha \tag{2.16}$$

with $0<\alpha\le\frac{1}{2}$. We close this section by prooving the following proposition 2.1.

PROPOSITION 2.1. – *If $\{X_n\}$ is a 1-dependent stationary sequence such that (2.16) is satisfied for some $\alpha>0$, then $\alpha\le\frac{1}{2}$.*

Proof. – The limit in (2.16) is α, $0\le\alpha\le 1$, if and only if

$$\lim_{u\to\omega}\frac{P\{X_1>u, X_2\le u\}}{P\{X_1>u\}}=1-\alpha.$$

Next, using 1-dependence and stationarity,

$$\begin{aligned}&\frac{P\{X_1>u, X_2>u, X_3>u\}}{P\{X_1>u\}}\\&=\frac{P\{X_1>u, X_2>u\}}{P\{X_1>u\}}-\frac{P\{X_1>u, X_2>u, X_3\le u\}}{P\{X_1>u\}}\\&\le\frac{P\{X_1>u, X_3>u\}}{P\{X_1>u\}}=P\{X_1>u\}\xrightarrow[u\to\omega]{}0.\end{aligned} \tag{2.17}$$

where

$$\frac{P\{X_1>u, X_2>u, X_3\le u\}}{P\{X_1>u\}}\le\frac{P\{X_1>u, X_2\le u\}}{P\{X_1>u\}}\xrightarrow[u\to\omega]{}1-\alpha. \tag{2.18}$$

But, by (2.17), the limit as $u\to\omega$ of the left hand term in (2.18) must be α. Thus $\alpha\le\frac{1}{2}$. ∎

3. EXTENSION OF THEOREM 1.1.

We now consider a 1-dependent sequence $\{X_n\}_{n\ge1}$ with continuous marginal d.f. $F(x)=P\{X_1\le x\}$, which we suppose strictly increasing near the right end point ω of X_1, i.e. for $x\ge x_0$, $x_0<\omega$. Furthermore, we suppose that $\{X_n\}_{n\ge1}$ satisfies the following hypotheses **H1** and **H2**:

H1.) The function $\tilde{H}(u)=P\{F(X_1)>u, F(X_2)>u\}$ is continuously differentiable near $u=1$ and

$$\tilde{H}'(1)=\lim_{x\to\omega}\frac{P\{X_1>x,\ X_2>x\}}{P\{X_1>x\}}=\alpha,\ 0\le\alpha\le 1/2. \tag{3.1}$$

H2.) There exists a constant $K > 0$ such that for $u \leq v \leq \omega$, we have

$$\begin{aligned} &P\{X_2 > u, \max(X_2, X_3) \in (v, v + dv)\} \\ &\quad \leq K\ P\{X_1 \leq u, X_2 > u, \max(X_2, X_3) \in (v, v + dv)\}. \end{aligned} \tag{3.2}$$

Remark 3.1. – It is rather easy to check, (even if for example 7 some calculations are quite lenghly) that all examples satisfy hypotheses **H1** and **H2**. If we compare condition (1.6) of theorem 1.1 and **H1**-**H2**, **H1** is weaker than (1.6) and then **H2** appears as a compensation of this fact.

Remark 3.2. – Consider the function

$$\begin{aligned} H(u) : &= P\{F(X_1) \leq u, F(X_2) > u\} \\ &= P\{F(X_1) > u\} - \tilde{H}(u) \\ &= 1 - u - \tilde{H}(u), \quad F(x_0) < u \leq 1. \end{aligned} \tag{3.3}$$

Hypothesis **H1** implies that there exists u_0, $F(x_0) \leq u_0 < 1$, such that $H(u)$ is strictly decreasing to zero for $u_0 \leq u \leq 1$. However, examples of 1-dependent stationary sequences for which such an u_0 does not exist (and thus **H1** is not satisfied) may be given. The following example has the form $X_n = f(\xi_n, \xi_{n+1})$, with $\{\xi_n\}_{n \geq 1}$ a sequence of i.i.d. r.v.'s. Observe that in this case, it is sufficient to construct $X_1 = f(U, V)$ and $X_2 = f(V, Z)$, U, V, Z i.i.d., in such a way that there exists an infinity of $u_n < v_n < \omega$, $u_n \nearrow \omega$ for which

$$P\{X_1 \leq u_n, X_2 > u_n\} < P\{X_1 \leq v_n, X_2 > v_n\}. \tag{3.4}$$

Let U have strictly positive density on $\mathbb{R}$ and let

$$X_1 = \begin{cases} U & \text{if } [V] = 2l + 1, \ l \in \mathbb{Z} \\ V + 1 & \text{if } [V] = 2l \end{cases}$$

and

$$X_2 = \begin{cases} V & \text{if } [Z] = 2l + 1, \ l \in \mathbb{Z} \\ Z + 1 & \text{if } [Z] = 2l \end{cases}$$

Observe that if $u_n < v_n$ satisfy (3.4), then

$$\begin{aligned} P\{X_1 \leq u_n, X_2 > u_n\} = &P\{X_1 \leq u_n, X_2 > v_n\} \\ &+ P\{X_1 \leq u_n, u_n < X_2 \leq v_n\} \end{aligned}$$

and

$$\begin{aligned} P\{X_1 \leq v_n, X_2 > v_n\} = &P\{X_1 \leq u_n, X_2 > v_n\} \\ &+ P\{u_n \leq X_1 \leq v_n, X_2 > v_n\}. \end{aligned}$$

whence (3.4) is equivalent to

$$P\{X_1 \le u_n, u_n < X_2 \le v_n\} < P\{u_n \le X_1 \le v_n, X_2 > v_n\}. \tag{3.5}$$

If we take $u_n = 2n$ and $v_n = 2n + 1$, then

$$P\{X_1 \le 2n, 2n < X_2 \le 2n + 1\} = 0$$

whereas

$$\begin{aligned} &P\{2n \le X_1 \le 2n + 1, X_2 > 2n + 1\} \\ &\quad \ge P\{[U] = 2n, [V] = 2n + 1, [Z] = 2n\} \\ &\quad = P\{[U] = 2n\} \times P\{[V] = 2n + 1\} \times P\{[Z] = 2n\} > 0. \end{aligned}$$

Thus, we have (3.5).

Let

$$h(u) := 1 - H(u). \tag{3.6}$$

The function $h(u)$ is strictly increasing to 1 for $u_0 \le u \le 1$ (see remark 3.2). We first state our results in terms of the 1-dependent stationary sequence

$$Z_n = h(F(X_n)), \quad n \ge 1, \tag{3.7}$$

whose right end point is 1.

DEFINITION 3.1. – *For any fixed ρ_0, $h(u_0) \le \rho_0 < 1$, define the "γ-2-block records" of $\{Z_n\}$, $\{(\tau_n, \rho_n)\}_{n \ge 1}$, as follows:*

$$\begin{aligned} \tau_1 &= \inf\{k \ge 1, Z_k > \rho_0\}, \\ \rho_1 &= \gamma_{\rho_0}(\max(Z_{\tau_1}, Z_{\tau_1+1})), \end{aligned} \tag{3.8}$$

where for any $\rho_0 \le \rho < z$,

$$\gamma_\rho(z) = z + P\{\xi_1 > \rho, \rho < \xi_2 \le z, \xi_3 > z\} \tag{3.9}$$

with $(\xi_1, \xi_2, \xi_3) \stackrel{\mathcal{L}}{=} (Z_1, Z_2, Z_3)$.

For $n \ge 1$

$$\tau_{n+1} = \inf\{k > \tau_n, \ Z_k > \rho_n\}$$

and (3.10)

$$\rho_{n+1} = \gamma_{\rho_n}(\max(Z_{\tau_n}, Z_{\tau_n - 1})).$$

Remark 3.3. – The sequence $\{(\tau_n, \rho_n)\}$ depends on the choice of the initial threshold ρ_0. However, we shall prove the following consistency result.

PROPOSITION 3.1. – *Let* $\{(\tau'_n, \rho'_n)\}$ *be a sequence such that for some* n_0, $\rho'_{n_0} \geq h(u_0)$ *and for* $n \geq n_0$, (τ'_n, ρ'_n) *are defined by the recurrence formulas (3.10), in which the* τ *and* ρ *are replaced by* τ' *and* ρ'.

Then there exists r.v.'s M *and* K *such that for* $n \geq M$,

$$\tau'_n = \tau_{n-K} \text{ and } \rho'_n = \rho_{n-K}. \tag{3.11}$$

Our main result is the following theorem.

THEOREM 3.1. – *Let* $\{X_n\}$ *satisfy hypotheses* **H1** *and* **H2**. *Then, there exists a probability space which carries, in addition to* $\{X_n\}$, *an i.i.d. uniformly on* $[0, 1]$ *distributed sequence* $\{\hat{Z}_n\}$. *Moreover, there exist two random variables* N *and* Q *such that a.s. for* $n \geq N$, *we have*

$$\tau_n = \hat{T}_{n-Q} \text{ and } \rho_n = \hat{\theta}_{n-Q}. \tag{3.12}$$

with $\{(\hat{T}_n, \hat{\theta}_n)\}$ *the record sequence of* $\{\hat{Z}_n\}$.

In terms of the initial sequence $\{X_n\}$, we then deduce the following corollary.

Consider the family of functions $\{G_u; u \in [F^{-1}(u_0), \omega]\}$, where $G_u : [u, \omega] \to [u, \omega]$ is defined as

$$G_u(v) = F^{-1} \circ h^{-1} \circ \gamma_{h \circ F(u)}(h \circ F(v)), \quad u < v < \omega, \tag{3.13}$$

with $\gamma_\rho(z)$ defined in (3.9) and u_0 of remark 3.2. We then have, for

$$F^{-1}(u_0) \leq u < \omega, \; G_u(u) = u \text{ and } G_u(v) \geq v, \; u \leq v < \omega.$$

For any fixed ρ_0^X, $F^{-1}(u_0) \leq \rho_0^X < \omega$, define the "G-2-block records" of $\{X_n\}$, $\{(\tau_n^X, \rho_n^X)\}_{n \geq 1}$, as follows:

$$\tau_1^X = \inf\{k \geq 1, X_k > \rho_0^X\},$$

$$\rho_1^X = G_{\rho_0^X}(\max(X_{\tau_1^X}, X_{\tau_1^X+1})).$$

and for $n \geq 1$ (3.14)

$$\tau_{n+1}^X = \inf\{k > \tau_n^X : X_k > \rho_n^X\},$$

and

$$\rho_{n+1}^X = G_{\rho_n^X}(\max(X_{\tau_n^X}, X_{\tau_n^X+1})).$$

COROLLARY 3.1. – *Let* $\{X_n\}$ *satisfy hypotheses* **H1** *and* **H2**. *Then, there exists a probability space which carries, in addition to* $\{X_n\}$ *an i.i.d. sequence* $\{\hat{X}_n\}_{n \geq 1}$ *with marginal distribution*

$$P\{\hat{X}_1 \leq x\} = 1 - P\{X_1 \leq x, X_2 > x\}, \quad x \geq F^{-1}(u_0).$$

Moreover, there exist random variables N and Q such that a.s. for $n \geq N$, we have

$$\tau_n^X = \hat{T}_{n-Q}^X \text{ and } \rho_n^X = \hat{\theta}_{n-Q}^X, \tag{3.15}$$

with $\{(\hat{T}_n^X, \hat{\theta}_n^X)\}$ the record sequence of $\{\hat{X}_n\}$.

DEFINITION 3.2. – *For any fixed $\mathcal{T}_0$, $h(u_0) \leq \mathcal{T}_0 < 1$, define the 2-block record sequence of $\{Z_n\}$ with respect to the initial threshold $\mathcal{T}_0$, $\{(\mathcal{T}_n, \mathcal{R}_n)\}_{n\geq 1}$, as*

$$\mathcal{T}_1 = \inf\{k \geq 1: Z_k > \mathcal{T}_0\}, \quad \mathcal{R}_1 = \max(Z_{\mathcal{T}_1}, Z_{\mathcal{T}_1+1})$$

and for $n \geq 1$

$$\mathcal{T}_{n+1} = \inf\{k > \mathcal{T}_n: Z_k > \mathcal{R}_n\}$$

and (3.16)

$$\mathcal{R}_{n+1} = \max(Z_{\mathcal{T}_{n+1}}, Z_{\mathcal{T}_{n+1}+1}).$$

$\mathcal{T}_n$ are the 2-block record times and $\mathcal{R}_n$ the 2-block record values.

Remark 3.4. – Let $\{(\mathcal{T}_n', \mathcal{R}_n')\}$ be a sequence such that for $n \geq n_0$, $\mathcal{T}_n'$ and $\mathcal{R}_n'$ are defined by the recurrence formulas (3.16), in which $\mathcal{T}$ and $\mathcal{R}$ are replaced by $\mathcal{T}'$ and $\mathcal{R}'$. Then, it can be easily seen that there exist m and k such that for $n \geq m$,

$$\mathcal{T}_n' = \mathcal{T}_{n-k} \text{ and } \mathcal{R}_n' = \mathcal{R}_{n-k}. \tag{3.17}$$

The next result is the following theorem 3.2.

THEOREM 3.2. – *If $\{X_n\}$ satisfies hypotheses* **H1** *and* **H2**, *then, there exist r.v.'s M and L such that a.s. for $n \geq M$, we have*

$$\tau_n = \mathcal{T}_{n-L} \text{ and } \mathcal{R}_{n-L} \leq \rho_n \leq \mathcal{R}_{n-L+1} \leq \rho_{n+1}. \tag{3.18}$$

The analogous result holds for the sequence of 2-block records of $\{X_n\}$, $\{(\mathcal{T}_n^X, \mathcal{R}_n^X)\}$ and $\{(\tau_n^X, \rho_n^X)\}$, i.e.

$$\tau_n^X = \mathcal{T}_{n-L}^X \text{ and } \mathcal{R}_{n-L}^X \leq \rho_n^X \leq \mathcal{R}_{n-L+1}^X.$$

Remark 3.5. – Combining (3.12) and (3.18), we also may say that there exist r.v.'s M' and L' such that a.s. for $n \geq M'$, we have

$$\hat{T}_n = \mathcal{T}_{n-L'} \text{ and } \mathcal{R}_{n-L'} \leq \hat{\theta}_n \leq \mathcal{R}_{n-L'+1} \leq \hat{\theta}_{n+1}.$$

This in particular means that the 2-block record times of $\{X_n\}$ (resp. $\{Z_n\}$) a.s. assymptotically coincide via an index translation to the record times of an i.i.d. sequence $\{\hat{X}_n\}$ (resp. $\{\hat{Z}_n\}$) and that the 2-block record values of $\{X_n\}$ (resp. $\{Z_n\}$) and the record values of $\{\hat{X}_n\}$ (resp. $\{\hat{Z}_n\}$) are imbricate.

In example 5 of section 2, we have $Z_n = 1 - X_n^2 + X_n^3 = X_n + O((1 - X_n)^2)$ and the 2-block records of $\{X_n\}$ a.s. coincide, via an index translation, to the records of the i.i.d. sequence $\{U_{n+1}\}$. This behaviour does not contradict the statements of Theorems 3.1 and 3.2.

We now prove proposition 3.1 and theorems 3.1 and 3.2 by means of the following intermediary results.

LEMMA 3.1. – *There exists a probability space which carries, in addition to $\{X_n\}$, a sequence $\{(S_n, R_n)\}_{n\geq 1}$ having the same Markov structure as the records of an i.i.d. sequence $\{\tilde{Z}_n\}$ with $U[0,1]$ distributed margins, i.e. for any $n \geq 0$,*
$1 \leq t_1 < t_2 < \ldots < t_n$, $s \geq 1$ *and* $s_0 = u_0 < u_1 < \ldots < u_{n+1} < 1$,

$$\begin{aligned}P\{S_{n+1} - S_n = s; R_{n+1} < u_{n+1} | S_1 = t_1, R_1 = u_1; \ldots; S_n = t_n, R_n = u_n\}\\ = P\{S_{n+1} - S_n = s; R_{n+1} < u_{n+1} | R_n = u_n\}\\ = u_n^{s-1}(u_{n+1} - u_n). \qquad (3.19)\end{aligned}$$

Furthermore, there exists a r.v. N such that a.s. for $n \geq 1$, S_n and R_n are defined by the recurrence formulas of the γ-2-block records, i.e.:

$$\begin{aligned}S_{n+1} = \inf\{k > S_n; Z_k > R_n\}\\ \text{and } R_{n+1} = \gamma_{R_n}(\max(R_{T_{n+1}}, R_{T_{n+1}+1})). \qquad (3.20)\end{aligned}$$

Remark 3.6. – Note that the sequence $\{(S_n, R_n)\}$ depends on the particular choice of the initial threshold u_0.

Proof. – By (Haiman (1987a), théorème 1', p. 432), there exist $0 < u_1 < 1$, a constant C_1 and a function $\mu(u)$, $0 < \mu(u) < 1$, such that

$$\begin{aligned}\max_{n\geq 1}\left|\frac{P\{\max(Z_1, \ldots, Z_n) \leq u\}}{\mu^n(u)} - 1 + P\{Z_1 > u, Z_2 > u\}\right|\\ \leq C_1\ (P(Z_1 > u))^2. \qquad (3.21)\end{aligned}$$

Furthermore, there exist constants C_2 and C_3 such that, for $u_1 \leq u \leq 1$, we have

$$\begin{aligned}|\mu(u) - (1 - P\{Z_1 \leq u, Z_2 > u\})|\\ = |\mu(u) - u| \leq C_2(P(Z_1 > u))^2 \leq C_3(1-u)^2. \qquad (3.22)\end{aligned}$$

Remark 3.7. – In the sequel we take for u_0 the maximum of u_0 in remark 3.2 and u_1.

For $k \geq 2$ and $u_0 \leq u < v < 1$, let

$$\begin{aligned}\mathcal{A}_k &= \mathcal{A}_k(u, dv)\\ &= \{Z_1 \leq u, \ldots, Z_{k-1} \leq u, Z_k > u, \gamma_u(\max(Z_k, Z_{k+1})) \in (v; v + dv]\}.\end{aligned} \tag{3.23}$$

It may be checked that

$$P\{\mathcal{A}_2(u, dv)\} = P\{Z_1 \leq u, Z_2 > u, \gamma_u(\max(Z_2, Z_3)) \in (v; v+dv]\} = dv. \tag{3.24}$$

Next,

$$\begin{aligned}&P\{\mathcal{A}_3\}\\ &= P\{\mathcal{A}_2\} - P\{Z_1 > u, Z_2 \leq u, Z_3 > u, \gamma_u(\max(Z_3, Z_4)) \in (v; v + dv]\},\end{aligned}$$

and, for $k \geq 4$,

$$\begin{aligned}P\{\mathcal{A}_k\} = {} & P\{Z_1 \leq u, \ldots, Z_{k-3} \leq u\} \cdot P\{\mathcal{A}_2\}\\ & - P\{Z_1 \leq u, \ldots, Z_{k-3} \leq u, Z_{k-2} > u, Z_{k-1} \leq u,\\ & Z_k > u, \gamma_u(\max(Z_k, Z_{k+1})) \in (v; v + dv]\},\end{aligned} \tag{3.25}$$

where, by stationarity and 1-dependence, the second term is majorizated by

$$\begin{aligned}&P\{Z_1 \leq u, \ldots, Z_{k-4} \leq u\} \cdot P\{Z_1 > u\}\\ &\quad \cdot P\{Z_1 > u, \gamma_u(\max(Z_1, Z_2)) \in (v; v + dv]\}\end{aligned} \tag{3.26}$$

(by convention, $P\{Z_0 \leq u\} = 1$). Now, by hypothesis **H2**,

$$P\{Z_1 > u, \gamma_u(\max(Z_1, Z_2)) \in (v; v + dv]\} = O_1(P\{\mathcal{A}_2\}), \quad u \to 1. \tag{3.27}$$

Thus, combining (3.21), (3.24)-(3.28), and observing that $u = 1 - (1 - u)$, we obtain, for $k \geq 2$,

$$P\{\mathcal{A}_k(u, dv)\} = u^{k-1} dv (1 + O_2((1 - u)^2))^{k-1} (1 + O_3(1 - u)), \tag{3.28}$$

where $|O_i(x)| \leq C_i\,|x|$ with C_i, $i = 1, 2, 3$ universal constants. We shall now construct $\{(S_n, R_n)\}$ by using recursively the following lemma.

LEMMA 3.2. (Haiman (1987a), p. 448). – *Let Y a random variable taking values in a measurable space $(\mathcal{Y}, \mathcal{F})$. Let $\varphi \in \mathcal{F}$ and let $\hat{P}$ be a probability* *measure on* $(\mathcal{Y}, \mathcal{F})$ *such that* $0 < \hat{P}(\varphi) < 1$. *Assume that on* φ, *the probability distribution of* Y, *denoted* P_Y, *has a Radon-Nicodym derivative* $\frac{dP_Y}{d\hat{P}}$ *with respect to* $\hat{P}$ *such that*

$$\max_{y \in \varphi} \left| \frac{dP_Y}{d\hat{P}}(y) - 1 \right| (1 - \hat{P}(\varphi))^{-1} \leq q < 1. \tag{3.29}$$

Let Q, *be a Bernoulli r.v. independent of* Y *such that* $P(Q = 0) = q$. *Then there exist two random variables* Y' *and* $\bar{Y}$, *taking values respectively in* $\mathcal{Y}$ *and* φ^c, *independent of* Y *and* Q, *and such that, if we put*

$$\hat{Y} = \begin{cases} Y & si\ Q = 1\ et\ Y \in \varphi \\ \bar{Y} & si\ Q = 1\ et\ Y \in \varphi^c \\ Y' & si\ Q = 0 \end{cases} \tag{3.30}$$

then, the probability distribution of $\hat{Y}$ *is* $\hat{P}$. *(*φ^c *denotes the complementary event of* φ.*)*

Let $S_0 = u_0$ and define (S_1, R_1) independently of $\{X_n\}_{n \geq 1}$. Moreover, suppose that for $1 \leq p \leq n$, (S_p, R_p) have allready been constructed and that the events of the form $\{(S_1 = s_1, R_1 \in \mathcal{A}_1), \ldots, (S_n = s_n, R_n \in \mathcal{A}_n)\}$ (where $1 \leq s_1 < \ldots < s_n$ and $\mathcal{A}_i, 1 \leq i \leq n$, are Borel sets of $\mathbb{R}$) are $\sigma\{X_1, \ldots, X_{s_n+1}\} \times \sigma'$ measurable, where σ' is a σ-field independent of $\sigma\{X_n, n \geq 1\}$.

Let $Y = Y_{s,r}$, $r \geq u_0$, be the random variable taking values in $\mathcal{Y} = \{1, 2, \ldots\} \times (r, 1)$, defined conditionaly given $S_n = s$ and $R_n = r$ as follows:

$$Y_{s,r} = (1, r'), r < r' < 1 \Leftrightarrow \{Z_{s+3} > r, \gamma_r(\max(Z_{s+3}, Z_{s+4})) = r'\} \tag{3.31}$$

and, for $t \geq 2$,

$$Y_{s,r} = (t, r'), r < r' < 1 \text{ if and only if}$$

$$\{Z_{s+3} \leq r, .., Z_{s+2+t-1} \leq r, Z_{s+2+t} > r, \gamma_r(\max(Z_{s+2+t}, Z_{s+3+t})) = r'\}. \tag{3.32}$$

Let $\hat{P}$, be the probability measure on $\mathcal{Y}$, defined conditionaly given $S_n = s$ and $R_n = r$ by

$$\hat{P}(t, [v, v + dv]) = r^{t-1} dv, \quad t \geq 1, r < v < 1. \tag{3.33}$$

Let

$$\varphi = \varphi_r = \{(t, r') \in \mathcal{Y}: 2 \leq t \leq [\psi(r)]\} \tag{3.34}$$

where

$$\psi(r) = \frac{\tau}{1-r} \ln_2\left(\frac{1}{1-r}\right), \; u_0 < r < 1, \tag{3.35}$$

with $\tau > 1$ a fixed constant and $\ln_2(x) = \ln(\ln x)$. We then have

$$\hat{P}(\varphi_r^c) = 1 - \hat{P}(\varphi) = 1 - r + r^{[\psi(r)]} > r^{[\psi(r)]} \sim \left(\ln\left(\frac{1}{1-r}\right)\right)^{-\tau}, \; r \to 1. \tag{3.36}$$

Next, denoting by Y_1, Y_2 the coordinates of $Y_{s,r} = Y$, observe that

$$P\{Y_1 = t, Y_2 \in [v, v + dv]\} = P\{\mathcal{A}_k(u, dv)\}. \tag{3.37}$$

Thus, combining (3.29), (3.32)-(3.38) and using the fact that $(1+x)^k = 1 + O(kx)$, $x \to 0$, we obtain

$$\max_{y \in \varphi_r} \left| \frac{dP_Y}{d\hat{P}}(y) - 1 \right| (1 - \hat{P}(\varphi))^{-1} \le c(1-r) \left(\ln\left(\frac{1}{1-r}\right)\right)^{\tau'} = q(r), \tag{3.38}$$

for some constants $C > 0$ and $\tau' > \tau$. Since $q(r) \to 0$ as $r \to 1$, for $r \ge u_2$ large enough (from now one we take for u_0 the maximum of u_0 in remark 3.7 and u_2), we may apply lemma 3.2.

Let $Q_{n+1} = Q$ with Q of the lemma (i.e. $P(Q_{n+1} = 0) = q(R_n)$).

Let L_{n+1} be a Bernoulli r.v. such that

$$P(L_{n+1} = 1) = r^2. \tag{3.39}$$

Furthermore, let Q_{n+1} and L_{n+1} be mutually independent and depend on $\{X_n\}$ and $(S_1, R_1), \ldots, (S_n, R_n)$ only through R_n.

Let $\hat{Y} = (\hat{Y}_1, \hat{Y}_2)$, be a random vector taking values in $\{1, 2\} \times (R_n, 1)$ such that

$$P\{\hat{Y}_1 = t; \hat{Y}_2 \le v\} = r^{t-1}(v - r)\frac{1}{P(L_{n+1} = 0)}, \; t = 1, 2, \; r < v < 1. \tag{3.40}$$

(We take $\hat{Y}$ independent of the previously defined random elements.) We now define (S_{n+1}, R_{n+1}) as follows:

If $L_{n+1} = 0$, then

$$S_{n+1} = S_n + \hat{Y}_1 \;\text{ and }\; R_{n+1} = \hat{Y}_2. \tag{3.41}$$

If $L_{n+1} = 1$, then

$$S_{n+1} = S_n + 2 + \hat{Y}_1 \;\text{ and }\; R_{n+1} = \hat{Y}_2. \tag{3.42}$$

where $(\hat{Y}_1, \hat{Y}_2) = \hat{Y}$ is given by lemma 3.2.

It is not difficult to check that $\{(S_1, R_1), \ldots, (S_{n+1}, R_{n+1})\}$ satisfy (3.19) and the measurability conditions requested previously at order n also at order $n+1$.

Now, in order to show that there exists a r.v. N such that a.s. for $n \geq N$, relation (3.20) is satisfied, it is sufficient to prove, denoting by E_n the event

$$E_n = \{L_{n+1} = 1, (\max(Z_{S_n+2}, Z_{S_n+3}) \leq R_n), Q_{n+1} = 1, Y \in \varphi_{R_n}\}, \tag{3.43}$$

that we have

$$P\{E_n^c \ \ i.o.\} = 0, \tag{3.44}$$

where E_n^c is the complementary event of E_n. In order to obtain (3.45), we will prove that

$$\begin{aligned} &P\{L_{n+1} = 0 \ i.o.\} = P\{\max(Z_{S_n+2}, Z_{S_n+3}) > R_n \ i.o.\} = \\ &P\{Q_{n+1} = 0 \ i.o.\} = P\{Y \in \varphi_{R_n}^c \ i.o.\} = 0. \end{aligned} \tag{3.45}$$

Remark 3.8. – Observe that if $\{(S_n, R_n)\}$ satisfies (3.19), then it is possible to construct an i.i.d. uniformly on $[0, 1]$ distributed sequence $\{\hat{Z}_n\}$ such that (S_n, R_n) is the record sequence of $\{\hat{Z}_n\}$ with respect to the initial threshold u_0.

Hence, by (Haiman (1987), Lemma 3, p. 454), for any $0 < \alpha < 1$, we have

$$P\{1 - R_n > e^{-\alpha n} \ \ i.o.\} = 0, \tag{3.46}$$

and for any $\beta > 1$,

$$P\{1 - R_n < e^{-\beta n} \ \ i.o.\} = 0. \tag{3.47}$$

Next, if A_n is one of the events in the brackets in (3.46), $P\{A_n \ i.o.\} = 0$ is equivalent to $P\{A_n \cap (1 - R_n \leq e^{-\alpha n}) \ i.o.\} = 0$.

Let

$$B_n = (L_{n+1} = 0) \cap (1 - R_n \leq e^{-\alpha n}). \tag{3.48}$$

We then have from (3.39)

$$P(B_n) = \int_{1-e^{-\alpha n}}^{1} (1 - r^2)\, dP_{R_n}(r) \leq c \ e^{-\alpha n},$$

with c positive constant, which is the general term of a convergent series. Thus

$$P(L_{n+1} = 0 \ \ i.o.) = 0$$

follows by the first Borel-Cantelli lemma.

In a similar way, we obtain

$$P\{(Q_{n+1} = 0) \cap (1 - R_n \leq e^{-\alpha n})\} \leq c' \ \ e^{-\alpha n}(\alpha n)^{\tau'}, \ 1 < \tau < \tau',$$

and

$$P\{(Y \in \varphi^c_{R_n}) \cap (1 - R_n \leq^{-\alpha n})\} \leq c'' \ n^{-\tau},$$

from which

$$P(Q_{n+1} = 0 \ \ i.o.) = P\{Y \in \varphi^c_{R_n} \ \ i.o.\} = 0$$

follows. Let now

$$C_n = (Q_{n+1} = 1) \cap (L_{n+1} = 1) \cap (Y \in \varphi^c_{R_n}) \cap (1 - R_n \leq e^{-\alpha n})$$

and

$$D_n = C_n \cap \{\max(Z_{S_n+2}, Z_{S_n+3}) > R_n\}.$$

We then similarly obtain

$$P(D_n) \leq c \, e^{-\alpha n} \ln(n)$$

and thus the last equality in (3.45). This achieves the proof of lemma 3.1. ■

Proof of Proposition 3.1 *and Theorems* 3.1 *and* 3.2. – In order to prove proposition 3.1 and theorems 3.1 and 3.2, we also need the following lemmas:

LEMMA 3.3. – *Consider, with* $\{(S_n, R_n)\}$ *of lemma 3.1, the events*

$$\begin{aligned}\mathcal{A}_n &= \{\inf[m > S_{n+1} + 2; Z_m > \max(Z_{S_{n+1}}, Z_{S_{n+1}+1})] < S_{n+2}\}\\ &= \bigcup_{m \geq 2} \{[\max(Z_{S_{n+1}+2}, \ldots, Z_{S_{n+1}+m}) \leq \max(Z_{S_{n+1}}, Z_{S_{n+1}+1})]\\ &\cap [Z_{S_{n+1}+m+1} \in (\max(Z_{S_{n+1}}, Z_{S_{n+1}+1}), \gamma_{R_n}(\max(Z_{S_{n+1}}, Z_{S_{n+1}+1}))]]\}.\end{aligned} \tag{3.49}$$

Then

$$P\{\mathcal{A}_n \ i.o.\} = 0. \tag{3.50}$$

Remark 3.9. – Lemma 3.3 implies that there exists a r.v. M such that a.s. for $n \geq M$,

$$S_{n+1} = \inf\{k > S_n; Z_k > \max(Z_{S_n}, Z_{S_n+1})\}. \tag{3.51}$$

i.e. for $n \geq M$ the S_n are given by the recurrence formulas of the 2-block record times of $\{Z_n\}$ in (3.16).

Proof of Lemma 3.3. – Let, taking into account (3.46) and (3.47),

$$\mathcal{B}_n = \mathcal{A}_n \cap (1 - R_n \leq^{-\alpha n}) \cap (1 - R_{n+1} > e^{-\beta n}).$$

We then have

$$P(\mathcal{B}_n) = \sum_{t=2n+1}^{\infty} \int\int_{\Delta} \times P\{\mathcal{A}_n | R_n = r, S_{n+1} = t, R'_{n+1} = s\} dP_{(R_n, S_{n+1}, R'_{n+1})}(r, t, s),$$

where

$$\Delta = \{(r, s); u_0 \leq r < s < 1,\ 1 - r \leq^{-\alpha n)},\ 1 - s > e^{-\beta n}\}$$

and

$$R'_{n+1} = \max(Z_{S_{n+1}}, Z_{S_{n+1}+1}).$$

Thus

$$\begin{aligned} &P\{\mathcal{A}_n R_n = r, S_{n+1} = t, R'_{n+1} = s\} \\ &\leq P\left\{ \bigcup_{m \geq 3} (\max(Z_{t+3}, \ldots, Z_{t+m}) < s) \cap (Z_{t+m+1} \in (s, \gamma_r(s)]) \right\} \\ &\leq K_1 P\{Z_1 \in (s, \gamma_r(s)]\} \sum_{k=0}^{\infty} \mu^k(s) \end{aligned}$$

with, by (3.22),

$$\mu(s) = s + O((1 - s)^2).$$

Thus, we easily deduce that there exists a positive constant K_2 such that

$$P\{\mathcal{A}_n | R_n = r, S_{n+1} = t, R'_{n+1} = s\} \leq K_2 \frac{\gamma_r(s) - s}{1 - s}.$$

Next, by (3.9)

$$\begin{aligned} \gamma_r(s) - s &= P\{\xi_1 > r, r < \xi_2 \leq s, \xi_3 > s\} \\ &\leq P\{\xi_1 > r\} \cdot P\{\xi_3 > s\} = O((1 - r)^2). \end{aligned} \tag{3.52}$$

Thus, there exists a constant K_3 such that on Δ

$$P\{\mathcal{A}_n | R_n = r, S_{n+1} = t, R'_{n+1} = s\} \leq K_3 \frac{e^{-2\alpha n}}{e^{-\beta n}} = K_3\, e^{-(2\alpha - \beta)n}.$$

Choose α and β such that $2\alpha - \beta > 0$. Then

$$P\{\mathcal{B}_n\} \leq K_3\, e^{-(2\alpha-\beta)n}$$

which is the general term of a convergent series and we get

$$P\{\mathcal{B}_n \ i.o.\} = 0.$$

whence (3.50). ■

Remark 3.10. – Let $\{(\tau'_n, \rho'_n)\}_{n\geq 1}$ be any sequence such that for $n \geq n_0$, (τ'_n, ρ'_n) are defined by the recurrence formulas of the γ-2-block records (3.10).

We then observe that either there exist N_1 and Q_1 such that for $n \geq N_1$

$$\tau'_{n-Q_1} = S_n \quad \text{and} \quad \rho'_{n-Q_1} = R_n.$$

or there exist N_2 and Q_2 such that for $n \geq N_2$

$$S_n < \tau'_{n-Q_2} < S_{n+1} \quad \text{and} \quad R_n < \rho'_{n-Q_2} < R_{n+1}.$$

Thus, by (3.46) and (3.47), we also have:
for any $0 < \alpha < 1$

$$P\{1 - \tau'_n > e^{-\alpha n} \ i.o.\} = 0$$

and for any $\beta > 1$

$$P\{1 - \tau'_n < e^{-\beta n} \ i.o.\} = 0.$$

Let $\mathcal{A}'_n$ be the event defined by formula (3.49), in which S_{n+1}, S_{n+2} are replaced by τ'_{n+1}, τ'_{n+2} and R_n by ρ'_n. We then may use the proof of lemma 3.3, without any other changes, to deduce that

$$P\{\mathcal{A}'_n \ i.o.\} = 0.$$

But this implies that there exists a r.v. M', such that a.s. for $n \geq M'$

$$\tau'_{n+1} = \inf\{k > \tau'_n ; Z_k > \max(Z_{\tau'_n}, Z_{\tau'_n+1})\}.$$

Then, by remark 3.4, there exist r.v.'s M' and K' such that a.s. for $n \geq M'$

$$\tau'_n = S_{n-K'} \quad \text{and} \quad \rho'_n = R_{n-K'}. \tag{3.53}$$

This proves proposition 3.1. The proof of theorem 3.1 also readily follows by taking $\{\hat{Z}_n\}$ the i.i.d. sequence of remark 3.8, $\hat{T}_n := S_n$ and $\hat{\theta}_n := R_n$. Remark 3.9 and theorem 3.1 then imply theorem 3.2. ■

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